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The **c**-2**d**-Index of Oriented Matroids

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We obtain an explicit method to compute the **cd**-index of the lattice of regions of an oriented matroid from the **ab**-index of the corresponding lattice of flats. Since the **cd**-index of the lattice of regions is a polynomial in the ring $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$, we call

ficient-wise. We give a new combinatorial description for the **c**-2**d**-index of the cubical lattice and the **cd**-index of the Boolean algebra in terms of all the permutations in the symmetric group S_n. Finally, we show that only two-thirds of the $\alpha(S)$'s of the lattice of flats are needed for the **c**-2**d**-index computation.

1. INTRODUCTION

The **cd**-index is an efficient way to encode the flag f-vector of a convex polytope. The generalized Dehn–Sommerville equations describe all the linear relations that hold among the entries of the flag f-vector, while the **cd**-index encodes the flag f-vector and removes the linear redundancies. For instance, the flag f-vector of a convex polytope of dimension n has 2^n entries, whereas the corresponding **cd**-index has only F_n entries. Here F_n is the nth Fibonacci number, where $F_0 = F_1 = 1$.

Originally suggested by Fine and developed by Bayer and Klapper [2], the **cd**-index is defined for all Eulerian posets. Recall that a poset is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x, y)}$. Observe that face lattices of convex polytopes are Eulerian posets.

Not very much is known about computing the **cd**-index. Purtill [21] gave recursion formulas for the **cd**-index of the Boolean algebra and the cubical lattice, that is, the face lattice of the cube. He also gave a combinatorial description of the coefficients of the **cd**-index of the Boolean

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algebra in terms of André permutations, a permutation class first studied by Foata and Schützenberger [12, 13]. Purtill also defined signed André permutations to obtain a similar result concerning the **cd**-index of the cubical lattice. Many authors have continued to work on understanding the **cd**-index of the simplex and the n-dimensional cube; see [10, 11, 16, 17].

The **cd**-index is also understood for simplicial polytopes, and more generally, for Eulerian simplicial posets. Stanley [24] expressed the **cd**-index of a simplicial polytope in terms of its h-vector and certain **cd**-polynomials $\check{\Phi}_i^n$. Hetyei [16] proved a conjecture of Stanley which gives a combinatorial interpretation of the **cd**-polynomials $\check{\Phi}_i^n$, whereas a short recursion for the $\check{\Phi}_i^n$ was found in [11]. Cubical polytopes, more generally Eulerian cubical posets, have been studied in [9].

In this paper we will consider oriented matroids. The lattice of regions of an oriented matroid is an Eulerian poset, thus it is natural to ask how to compute its **cd**-index. We provide here an answer to this question.

The lattice of flats of a matroid describes the combinatorial structure of the matroid. Zaslavsky [30] showed that the lattice of flats of an oriented matroid completely determines the f-vector of the lattice of regions, while Bayer and Sturmfels [3] showed that it completely determines the flag f-vector. Our work describes this relation explicitly. Namely, let $\Psi(L)$ be the **ab**-index of the lattice of flats. We compute the **cd**-index of the lattice of regions by replacing every occurrence of **ab** in $\mathbf{a} \cdot \Psi(L)$ by $2\mathbf{d}$ and replacing each of the remaining letters by **c**. Observe that every **d** in the **cd**-index has a factor of 2 associated with it. Thus the **cd**-index is naturally written in terms of **c** and $2\mathbf{d}$, and hence it is called the **c**-2**d**-index.

The proof of the main theorem is based upon recasting in terms of **ab**-indexes a result relating the number of chains in the lattice of regions of an oriented matroid to the Möbius function of its associated lattice of flats (see [7, Proposition 4.6.2] or Proposition 6.1). In order to apply Proposition 6.1 we need to use the fact that the **ab**-index is a coalgebra homomorphism; see [11]. We review the coalgebra techniques of [11] in Section 4 and then develop the necessary tools in Section 5 so that we can interpret the identity in Proposition 6.1 in terms of the **ab**-index. By these techniques we obtain the explicit relation between the **ab**-index of the lattice of flats and the **c**-2**d**-index of the lattice of regions.

A special class of oriented matroids are realizable oriented matroids. They correspond to hyperplane arrangements in the sense that the lattice of regions of a realizable oriented matroid is isomorphic to the face lattice of its corresponding hyperplane arrangement. Every hyperplane arrangement has a corresponding zonotope, and the lattice of faces of the hyperplane arrangement is anti-isomorphic to the face lattice of this zonotope. Hence we have a method to compute the **c**-2**d**-index of a zonotope.

It was conjectured by Stanley [25] that coefficient-wise the **cd**-index of an n-dimensional convex polytope is greater than or equal to the **cd**-index of the n-simplex. More generally, he conjectured that the **cd**-index of a Gorenstein* lattice of rank n is greater than or equal to the **cd**-index of the Boolean algebra of rank n. As a corollary of our result we obtain the zonotopal analogue of this conjecture: the **c**-2**d**-index of the lattice of regions of an oriented matroid of rank n is coefficient-wise greater than or equal to the **c**-2**d**-index of the arrangement consisting of the coordinate hyperplanes. That is, among all zonotopes of dimension n, the n-dimensional cube has the smallest **c**-2**d**-index.

As one easy application of our main result, we obtain a natural way to compute the **c**-2**d**-index of the cubical lattice in terms of all permutations in the symmetric group S_n. This avoids having to use restricted classes of permutations, such as signed André permutations and signed simsun permutations. As a consequence, we find a straightforward way to compute the **cd**-index of the Boolean algebra in terms of all permutations in the symmetric group.

Finally, in the last section we show that not all of the $\alpha(S)$'s of the lattice of flats are needed to compute the **c**-2**d**-index of the oriented matroid. Surprisingly we only need two-thirds of the $\alpha(S)$'s. We give an explicit description of this essential set of $\alpha(S)$'s.

2. DEFINITIONS

In this paper we will consider graded posets of rank greater than or equal to one, that is, posets P having a minimal element $\hat{0}$ and a maximal element $\hat{1}$ so that $\hat{0} \neq \hat{1}$. Moreover, there is a rank function ρ such that $\rho(\hat{0}) = 0$. For $x \leqslant y$ define $\rho(x, y)$ to be equal to $\rho(y) - \rho(x)$ and the *interval* $[x, y]$ to be the set $\{z : x \leqslant z \leqslant y\}$. Observe that $[x, y]$ is a graded poset of rank $\rho(x, y)$.

A poset L is a *lattice* if every pair of elements x, y has a unique greatest lower bound $x \wedge y$, called the *meet*, and a unique least upper bound $x \vee y$, called the *join*. A ranked lattice L is called *semi-modular* if it satisfies the following inequality:

$$\rho(x) + \rho(y) \geqslant \rho(x \wedge y) + \rho(x \vee y),$$

for all $x, y \in L$. A lattice is *atomic* if all of its elements can be written as a join of atoms. A lattice which is both semi-modular and atomic is a *geometric lattice*.

The *Möbius function* $\mu(x, y)$ is defined for $x, y \in P$ such that $x \leqslant y$ by $\mu(x, x) = 1$ and $\sum_{x \leqslant z \leqslant y} \mu(x, z) = 0$ for $x < y$. We denote $\mu(\hat{0}, \hat{1})$ by $\mu(P)$.

Let P be a graded poset of rank $n+1$. For S a subset of $\{1, ..., n\}$ let P_S be the subposet of P defined as $P_S = \{x \in P: \rho(x) \in S,\ x = \hat{0},\ \text{or}\ x = \hat{1}\}$. Let $\alpha(S)$ be the number of maximal chains in P_S. That is, $\alpha(S)$ is the number of chains in P whose ranks correspond to the set S. Define $\beta(S)$ by the equation

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \alpha(T). \tag{1}$$

Then we have that $\beta(S) = (-1)^{|S|+1} \cdot \mu(P_S)$; see Equation (34) in [23, Section 3.12].

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutating variables. For a subset S of $\{1, ..., n\}$ define u_S to be the **ab**-monomial $u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* $\Psi(P)$ of a poset P of rank $n+1$ is defined by

$$\Psi(P) = \sum_S \beta(S) \cdot u_S, \tag{2}$$

where the sum ranges over all subsets S of $\{1, ..., n\}$. Observe that $\Psi(P)$ is a homogeneous polynomial of degree n.

An alternative definition of the **ab**-index is given by assigning weights to each chain in P. For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ define the *weight* of the chain c to be the product $\mathrm{wt}(c) = w_1 \cdots w_n$, where

$$w_i = \begin{cases} \mathbf{b} & \text{if} \quad i \in \{\rho(x_1), ..., \rho(x_{k-1})\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

Hence the weight of the chain c is given by

$$\mathrm{wt}(c) = (\mathbf{a} - \mathbf{b})^{\rho(x_0, x_1) - 1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_1, x_2) - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_{k-1}, x_k) - 1}.$$

Then the **ab**-index is given by

$$\Psi(P) = \sum_c \mathrm{wt}(c), \tag{3}$$

where c ranges over all chains c in the poset P.

A poset P is called *Eulerian* if the Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x, y)}$. When P is Eulerian the **ab**-index of P can be written in terms of the non-commutating variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. This observation is due to Fine, see [2]; for an elementary proof, see [24]. In [4] it was observed that when P is the lattice of regions of an oriented matroid $\mathscr{M}$, then the **ab**-index of P can be written as a polynomial with integer coefficients in the non-commuting variables $\mathbf{c}$ and $2 \cdot \mathbf{d}$. When $\Psi(P)$ is written in terms of $\mathbf{c}$ and $2\mathbf{d}$, we call $\Psi(P)$ the **c-2d**-*index*.

Let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ be the ring of polynomials in the variables **a** and **b**, and let the degree of **a** and **b** be 1. Let $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$ denote the subring of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ spanned by the elements $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $2\mathbf{d} = 2\mathbf{ab} + 2\mathbf{ba}$ and let $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle^+$ denote **c**-2**d**-polynomials without constant coefficient.

For a poset P let P^* denote the *dual* poset. The poset P^* has the same underlying set as P but with the order relation $x \leqslant_{P^*} y$ if $x \geqslant_P y$. Similarly for an **ab**-monomial $v = v_1 v_2 \cdots v_n$ let $v^* = v_n \cdots v_2 v_1$. By linearity we extend this operation to be an involution on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$. Since $\mathbf{c}^* = \mathbf{c}$ and $2\mathbf{d}^* = 2\mathbf{d}$, the involution restricts to $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$ by reading the **c**-2**d**-monomial backwards. Observe we have for a graded poset P that $\Psi(P^*) = \Psi(P)^*$.

The important function we will work with is ω, which we now describe.

DEFINITION 2.1. Define a linear function $\omega: \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$ as follows: For an **ab**-monomial v we compute $\omega(v)$ by replacing each occurrence of **ab** in the monomial v with 2**d**, then replacing the remaining letters with **c**'s. Extend this definition by linearity to **ab**-polynomials.

The function ω takes an **ab**-polynomial of degree n into a **c**-2**d**-polynomial of degree n. As an example

$$\begin{aligned}\omega(\mathbf{aaa} + 3\cdot\mathbf{aba} + 5\cdot\mathbf{aab} + 3\cdot\mathbf{abb}) &= \mathbf{c}^3 + 3\cdot 2\mathbf{d}\cdot\mathbf{c} + 5\cdot\mathbf{c}\cdot 2\mathbf{d} + 3\cdot 2\mathbf{d}\cdot\mathbf{c}\\ &= \mathbf{c}^3 + 6\cdot 2\mathbf{d}\cdot\mathbf{c} + 5\cdot\mathbf{c}\cdot 2\mathbf{d}.\end{aligned}$$

3. THE MAIN THEOREM FOR ORIENTED MATROIDS

Form a poset from the set $\{+, -, 0\}$ by the order relations $0 < +$, $0 < -$, and $+$ and $-$ are incomparable. If E is a finite set then the set $\{+, -, 0\}^E$ is also a poset. Observe that this poset does not have a maximal element. An element of $\{+, -, 0\}^E$ is called a *sign vector*.

An *oriented matroid* $\mathcal{M}$ on the set E is a collection of sign vectors from $\{+, -, 0\}^E$, called covectors, which satisfies the covector axioms of an oriented matroid; see [7, Definition 4.1.1]. We refer the reader to [7, Chapter 4] for more details on oriented matroids. Observe that the covectors form a subposet of the poset $\{+, -, 0\}^E$. Let T denote the poset of covectors with a maximal element $\hat{1}$ adjoined. The poset T is a lattice which we call the *lattice of regions*. The coatoms of T correspond to the topes in the hyperplane arrangement; hence we use the notation T to denote this lattice.

The lattice of regions of an oriented matroid is isomorphic to the face lattice of a regular cell decomposition of a sphere of dimension $\rho(T) - 2$. In fact, an interval $[x, y]$ in the lattice of regions is isomorphic to the face lattice of a regular cell decomposition of a sphere of dimension $\rho(x, y) - 2$.

Moreover, these cell decompositions are shellable. See Theorem 4.3.3 and Corollary 4.3.7 in [7] for more details. These results imply that the lattice of regions of an oriented matroid is an Eulerian poset; see [7, Corollary 4.3.8].

Underlying every oriented matroid there is a matroid. Any matroid can be described by its *lattice of flats*, which is a geometric lattice. We denote the lattice of flats of an oriented matroid $\mathscr{M}$ by L. It follows from Bayer and Sturmfels [3, Theorem 3.4] that the **ab**-index of the lattice of regions T depends only on the lattice L. Our main theorem will show this dependency in an explicit manner.

As an example of an oriented matroid, consider a linear hyperplane arrangement $\mathscr{H} = \{H_e\}_{e \in E}$ in $\mathbb{R}^n$. Assume that $\bigcap_{e \in E} H_e = \{0\}$, that is, the collection $\mathscr{H}$ is *essential*. For each hyperplane H_e choose a normal vector $\mathbf{u}_e$. The hyperplane arrangement $\mathscr{H}$ cuts $\mathbb{R}^n$ into cones. Let a partial order on the set of cones be given by the cone C is less than or equal to the cone C' if the closure of C is contained in the closure of C'. If we adjoin a maximal element to this poset then it is isomorphic to the lattice of regions of the corresponding oriented matroid $\mathscr{M}$. A cone C corresponds to the sign vector x if $x_e = \operatorname{sign}(\mathbf{u}_e \cdot \mathbf{x})$ for a vector $\mathbf{x}$ in the relative interior of the cone C.

The intersection lattice of the hyperplane arrangement $\mathscr{H}$ is the lattice on the set of subspaces $\{\bigcap_{e \in S} H_e : S \subseteq E\}$ ordered by reversed inclusion. Thus $\mathbb{R}^n$ is the minimal element and $\{0\}$ is the maximal element; the hyperplanes in the arrangement are the atoms. Note the intersection lattice of the hyperplane arrangement $\mathscr{H}$ is isomorphic to the lattice of flats L of the underlying matroid $\mathscr{M}$.

Associated to the essential hyperplane arrangement $\mathscr{H}$ is an n-dimensional zonotope Z, the Minkowski sum of the normals to the hyperplanes in $\mathscr{H}$; see [7]. The face lattice $\mathscr{L}(Z)$ of the zonotope Z is anti-isomorphic to the lattice of regions of the corresponding oriented matroid $\mathscr{M}$, that is, $\mathscr{L}(Z) = T^*$.

Since the lattice of regions T is an Eulerian poset, it has a **cd**-index. In fact, the lattice T has a **c**-2**d**-index and the following theorem shows how to compute its **c**-2**d**-index.

THEOREM 3.1. *Let $\mathscr{M}$ be an oriented matroid, T the lattice of regions of $\mathscr{M}$, and L the lattice of flats of $\mathscr{M}$. Then the* **c**-2**d**-*index of T is given by*

$$\Psi(T) = \omega(\mathbf{a} \cdot \Psi(L))^*.$$

For example, consider the hyperplane arrangement in $\mathbb{R}^3$ with the four hyperplanes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 0$. The corresponding intersection lattice L has rank 3, and it is straightforward to compute its **ab**-index,

$$\begin{aligned}\Psi(L) &= \alpha(\emptyset)\cdot\mathbf{aa} + (\alpha(1)-\alpha(\emptyset))\cdot\mathbf{ba} + (\alpha(2)-\alpha(\emptyset))\cdot\mathbf{ab}\\ &\quad + (\alpha(1,2)-\alpha(1)-\alpha(2)+\alpha(\emptyset))\cdot\mathbf{bb}\\ &= \mathbf{aa} + 3\cdot\mathbf{ba} + 5\cdot\mathbf{ab} + 3\cdot\mathbf{bb}.\end{aligned}$$

(Note that throughout we will omit the brackets in expressions involving α's and β's.) Hence we obtain the **c**-2**d**-index of the lattice of regions by

$$\begin{aligned}\omega(\mathbf{aaa} + 3\cdot\mathbf{aba} + 5\cdot\mathbf{aab} + 3\cdot\mathbf{abb})^* &= (\mathbf{c}^3 + 6\cdot 2\mathbf{d}\cdot\mathbf{c} + 5\cdot\mathbf{c}\cdot 2\mathbf{d})^*\\ &= \mathbf{c}^3 + 6\cdot\mathbf{c}\cdot 2\mathbf{d} + 5\cdot 2\mathbf{d}\cdot\mathbf{c}.\end{aligned}$$

As a consequence of Theorem 3.1 we have the following four corollaries.

COROLLARY 3.2. *Let $\mathscr{H}$ be an essential hyperplane arrangement and let L be the intersection lattice of $\mathscr{H}$. Let Z be the zonotope that corresponds to $\mathscr{H}$. Then the* **c**-2**d**-*index of the zonotope Z is given by*

$$\Psi(Z) = \omega(\mathbf{a}\cdot\Psi(L)).$$

Since the sum of the coefficients of the **ab**-index of a poset P is the number of maximal chains of P, we obtain the following corollary.

COROLLARY 3.3. *The sum of the coefficients of the* **c**-2**d**-*index of the lattice of regions is equal to the number of maximal chains in the lattice of flats.*

The rank of the oriented matroid is defined to be the rank of the lattice of regions minus one, that is, $\rho(T)-1$.

COROLLARY 3.4. *Let $\mathscr{M}$ be an oriented matroid of even rank n. Then the coefficient of $(2\mathbf{d})^{n/2}$ in the* **c**-2**d**-*index of the lattice of regions is given by the value $\beta(1, 3, ..., n-1)$ of the lattice of flats L.*

One may also use Theorem 3.1 to find an expression for the f-vector of a zonotope in terms of the β-invariant of the corresponding lattice of flats, obtaining an equivalent version of Zaslavsky's Corollary 5.5 [30] (which is in terms of the α invariant).

COROLLARY 3.5. *For an oriented matroid $\mathscr{M}$ of rank $n+1$, let $f_k(T)$ denote the number of k-dimensional faces in the dual to the associated lattice* *of regions T (the dual $(n+1)$-dimensional zonotope, in the realizable case). Then for $0 \leqslant k \leqslant n$,*

$$f_k(T) = 2\left(1 + \sum_{m=k+1}^{n} \beta(k+1, \ldots, m) + \sum_{m=k}^{n} \beta(k, \ldots, m)\right),$$

where $\beta = \beta_L$, the β-invariant of the corresponding lattice of flats, and where we take $\beta(0, \ldots, m) = 0$.

4. COALGEBRA TECHNIQUES

In order to prove our main theorem we will be using the fact that the **ab**-index may be viewed as a coalgebra homomorphism. We develop briefly this idea in this section. For greater detail we refer the reader to [11].

Let $\mathscr{P}$ denote the integer span of the set of all isomorphism types of graded posets of rank greater than or equal to 1. On the space $\mathscr{P}$ introduce a coproduct by

$$\Delta(P) = \sum_{\hat{0} < x < \hat{1}} [\hat{0}, x] \otimes [x, \hat{1}].$$

The Sweedler notation of coproducts is to write $\Delta(P) = \sum_P P_{(1)} \otimes P_{(2)}$; see [28]. This coproduct is coassociative, hence the linear map $\Delta^{k-1}: \mathscr{P} \to \mathscr{P}^{\otimes k}$ is defined by

$$\Delta^{k-1}(P) = \sum_{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}} [x_0, x_1] \otimes [x_1, x_2] \otimes \cdots \otimes [x_{k-1}, x_k].$$

Similarly, the Sweedler notation for Δ^{k-1} is $\Delta^{k-1}(P) = \sum_P P_{(1)} \otimes \cdots \otimes P_{(k)}$.

There is a natural coproduct on the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. For a monomial $v = v_1 \cdots v_n$ let

$$\Delta(v) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n,$$

and extend Δ to the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by linearity. The coproduct Δ on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is coassociative. When the degree of a monomial v is less than k, we have that $\Delta^k(v) = 0$. Also observe that Δ satisfies the Newtonian condition [18]:

$$\Delta(u \cdot v) = \sum_u u_{(1)} \otimes u_{(2)} \cdot v + \sum_v u \cdot v_{(1)} \otimes v_{(2)}. \tag{4}$$

We remark that neither $\mathscr{P}$ nor $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ has an augmentation (counit). Hence they are not coalgebras in the classical sense.

We may extend the map Ψ by linearity to the space $\mathscr{P}$. From [11, Proposition 3.1] we have the following proposition.

PROPOSITION 4.1. ([11]). *The linear map $\Psi: \mathscr{P} \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is a coalgebra homormophism, that is, for a poset P we have*

$$\Delta(\Psi(P)) = \sum_{P} \Psi(P_{(1)}) \otimes \Psi(P_{(2)}).$$

This result is important since it allows us to obtain information about the **ab**-index of intervals $[\hat{0}, x]$ and $[x, \hat{1}]$ by knowing only the **ab**-index of the entire poset. More generally, let $f_1, ..., f_k$ be linear maps on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. Then by coassociativity of the coproduct and Proposition 4.1 we may compute

$$\sum_{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}} f_1(\Psi([x_0, x_1])) \cdots f_k(\Psi([x_{k-1}, x_k]))$$

by knowing only the **ab**-index $\Psi(P)$ of the entire poset P, rather than the **ab**-index of each of the intervals in the poset. That is, the previous expression is equal to

$$\sum_{v} f_1(v_{(1)}) \cdots f_k(v_{(k)}),$$

where $v = \Psi(P)$.

5. THE THREE FUNCTIONS κ, η, AND φ

We will now introduce certain linear functions on the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. These functions will be important in giving Proposition 6.1 an interpretation in terms of the **ab**-index, which will imply our main theorem. We begin by defining two linear functions $A, B: \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}$ by defining them on an **ab**-monomial v by

$$A(v) = \begin{cases} 1 & \text{if} \quad v = \mathbf{a}^k \quad \text{for some} \quad k \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B(v) = \begin{cases} 1 & \text{if} \quad v = \mathbf{b}^m \quad \text{for some} \quad m \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that A and B are ring homomorphisms. Next, define the linear function $E: \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}$ by

$$E(v) = \begin{cases} 1 & \text{if} \quad v = \mathbf{b}^m \mathbf{a}^k \quad \text{for some} \quad m, k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

The three functions A, B, and E are related by the following lemma.

LEMMA 5.1. *For all elements $v \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ we have*

$$2 \cdot E(v) = A(v) + B(v) + \sum_v B(v_{(1)}) \cdot A(v_{(2)}).$$

Proof. Since both sides are linear in v, it is enough to prove it for an **ab**-monomial v. Indeed, if v is of the form $u \cdot \mathbf{ab} \cdot u'$ then both sides are equal to zero. On the other hand, for v of the form $\mathbf{b}^m \mathbf{a}^k$, it reduces to checking that $2 = \chi(k=0) + \chi(m=0) + \chi(k>0) + \chi(m>0)$. ∎

Note that for a graded poset P we have that $A(\Psi(P)) = 1$ and $B(\Psi(P)) = (-1)^{\rho(P)} \cdot \mu(P)$.

LEMMA 5.2. *Let P be a graded poset. Then*

$$\sum_{\hat{0} \leqslant x \leqslant \hat{1}} (-1)^{\rho(x)} \cdot \mu(\hat{0}, x) = 2 \cdot E(\Psi(P)).$$

Proof. Let v be the **ab**-index of the poset P, that is, $v = \Psi(P)$. Then we obtain

$$\begin{aligned} \sum_{\hat{0} \leqslant x \leqslant \hat{1}} (-1)^{\rho(x)} \cdot \mu(\hat{0}, x) &= 1 + (-1)^{\rho(P)} \cdot \mu(P) + \sum_{\hat{0} < x < \hat{1}} (-1)^{\rho(x)} \cdot \mu(\hat{0}, x) \\ &= A(\Psi(P)) + B(\Psi(P)) + \sum_{\hat{0} < x < \hat{1}} B(\Psi([\hat{0}, x])) \\ &= A(\Psi(P)) + B(\Psi(P)) + \sum_P B(\Psi(P_{(1)})) \cdot A(\Psi(P_{(2)})) \\ &= A(v) + B(v) + \sum_v B(v_{(1)}) \cdot A(v_{(2)}) \\ &= 2 \cdot E(v). \quad ∎ \end{aligned}$$

If P is a poset of rank $n+1$, notice that

$$E(\Psi(P)) = \sum_{i=0}^{n} \beta_P(1, ..., i).$$

In terms of the α's we may write this as

$$E(\Psi(P)) = \sum_{S} (-1)^{n-|S|} \cdot \alpha(S),$$

where S ranges over all subsets of $\{1, ..., n\}$ such that $\max(S \cup \{0\}) \equiv n \bmod 2$.

As a side remark, we may compute the *characteristic polynomial* $\chi(P; q)$ of a poset P using a similar technique. Recall that the characteristic polynomial is defined by

$$\chi(P; q) = \sum_{\hat{0} \leqslant x \leqslant \hat{1}} \mu(\hat{0}, x) \cdot q^{\rho(x, \hat{1})}.$$

Let $\bar{E}\colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}[q]$ be given by

$$\bar{E}(v) = \begin{cases} (-1)^m \cdot q^k & \text{if} \quad v = \mathbf{b}^m \mathbf{a}^k \quad \text{for some} \quad m, k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by a proof similar to that of Lemma 5.2 (with $\bar{A}(\mathbf{a}^k) = q^{k+1}$, $\bar{B}(\mathbf{b}^m) = (-1)^{m+1}$ and $(q-1) \cdot \bar{E}(v) = \bar{A}(v) + \bar{B}(v) + \sum_v \bar{B}(v_{(1)}) \cdot \bar{A}(v_{(2)})$, we can obtain

PROPOSITION 5.3. *The characteristic polynomial of a poset P is related to its* **ab**-*index by*

$$\chi(P; q) = (q-1) \cdot \bar{E}(\Psi(P)).$$

We now define three linear functions κ, η, and φ from $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ that will be very useful for us. The linear function κ is defined on **ab**-monomials by

$$\kappa(v) = \begin{cases} (\mathbf{a} - \mathbf{b})^k & \text{if} \quad v = \mathbf{a}^k \quad \text{for some} \quad k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, η is defined on **ab**-monomials by

$$\eta(v) = \begin{cases} 2 \cdot (\mathbf{a} - \mathbf{b})^{m+k} & \text{if} \quad v = \mathbf{b}^m \mathbf{a}^k \quad \text{for some} \quad m, k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the functions κ and η correspond to the functions $A(v)$ and $2 \cdot E(v)$ respectively, that is, we have the relations $\kappa(v) = A(v) \cdot (\mathbf{a} - \mathbf{b})^{\deg(v)}$ and $\eta(v) = 2 \cdot E(v) \cdot (\mathbf{a} - \mathbf{b})^{\deg(v)}$. Hence for a poset P we have

$$\kappa(\Psi(P)) = A(\Psi(P)) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}, \tag{5}$$

$$\eta(\Psi(P)) = 2 \cdot E(\Psi(P)) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}. \tag{6}$$

To define the third function φ, we begin to define $\varphi_k : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ for $k \geqslant 1$ by

$$\varphi_k(v) = \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k)}).$$

When $\deg(v) < k-1$, we obtain $\varphi_k(v) = 0$. Hence the sum $\sum_{k \geqslant 1} \varphi_k(v)$ is always finite, so we define $\varphi(v) = \sum_{k \geqslant 1} \varphi_k(v)$. Notice that the three linear maps κ, η, and φ are all degree-preserving, that is, for a monomial v of degree n, we have $\kappa(v)$, $\eta(v)$, and $\varphi(v)$ are homogeneous of degree n. Moreover, these three maps satisfy the following functional equation.

Lemma 5.4.

$$\varphi(v) = \kappa(v) + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}).$$

Proof. We have $\varphi_1(v) = \kappa(v)$. Since the coproduct is coassociative, we obtain for $k \geqslant 2$ that

$$\begin{aligned} \varphi_k(v) &= \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k-1)}) \cdot \mathbf{b} \cdot \eta(v_{(k)}) \\ &= \sum_v \left(\sum_{v_{(1)}} \kappa(v_{(1,1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(1,k-1)}) \right) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \\ &= \sum_v \varphi_{k-1}(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}). \end{aligned}$$

Now summing over $k \geqslant 1$ we get the desired equality. ∎

Lemma 5.4 enables us to compute values of the function φ. For instance, we have that $\varphi(1) = 1$, $\varphi(\mathbf{a}) = \mathbf{c}$, and $\varphi(\mathbf{a} \cdot \mathbf{b}) = 2\mathbf{d}$ since $\varphi(\mathbf{a} \cdot \mathbf{b}) = \kappa(\mathbf{a} \cdot \mathbf{b}) + \varphi(\mathbf{a}) \cdot \mathbf{b} \cdot \eta(1) + \varphi(1) \cdot \mathbf{b} \cdot \eta(\mathbf{b}) = 0 + \mathbf{c} \cdot \mathbf{b} \cdot 2 + 1 \cdot \mathbf{b} \cdot 2 \cdot (\mathbf{a} - \mathbf{b}) = 2\mathbf{d}$.

The main result of this section is that the function φ is related to the function ω (see Definition 2.1) in the following manner.

Proposition 5.5. *For an* **ab**-*monomial v which begins with the letter* **a** *we have*

$$\omega(v) = \varphi(v).$$

Observe that the proposition does not hold for monomials that begin with the letter $\mathbf{b}$. For instance, $\omega(\mathbf{b}) = \mathbf{c}$, but $\varphi(\mathbf{b}) = 2 \cdot \mathbf{b}$.

The proof of this proposition will follow with the help of the following two lemmas.

LEMMA 5.6. *Let v be a nonconstant* **ab**-*monomial and let x be either* **a** *or* **b**. *Assume that the monomial $v \cdot x$ does not end with* **ab**. *Then*

$$\varphi(v \cdot x) = \varphi(v) \cdot \mathbf{c}.$$

Proof. The conditions in the statement say that either $x = \mathbf{a}$ or that $x = \mathbf{b}$ and v ends with $\mathbf{b}$. In both cases it is easy to check that the two equalities $\kappa(v \cdot x) = \kappa(v) \cdot (\mathbf{a} - \mathbf{b})$ and $\eta(v \cdot x) = \eta(v) \cdot (\mathbf{a} - \mathbf{b})$ hold.

By the Newtonian condition (4) we have

$$\Delta(v \cdot x) = v \otimes 1 + \sum_v v_{(1)} \otimes v_{(2)} \cdot x.$$

Hence by Lemma 5.4 we obtain that

$$\begin{aligned}
\varphi(v \cdot x) &= \kappa(v \cdot x) + \varphi(v) \cdot \mathbf{b} \cdot \eta(1) + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)} \cdot x) \\
&= \kappa(v) \cdot (\mathbf{a} - \mathbf{b}) + \varphi(v) \cdot \mathbf{b} \cdot 2 + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \varphi(v) \cdot (\mathbf{a} - \mathbf{b}) + \varphi(v) \cdot \mathbf{b} \cdot 2 \\
&= \varphi(v) \cdot (\mathbf{a} + \mathbf{b}) \\
&= \varphi(v) \cdot \mathbf{c}. \quad \blacksquare
\end{aligned}$$

LEMMA 5.7. *Let v be a nonconstant* **ab**-*monomial. Then*

$$\varphi(v \cdot \mathbf{a} \cdot \mathbf{b}) = \varphi(v) \cdot 2\mathbf{d}.$$

Proof. Observe that $\kappa(v \cdot \mathbf{a} \cdot \mathbf{b}) = 0$ and $\eta(v \cdot \mathbf{a} \cdot \mathbf{b}) = 0$. By the Newtonian condition (4) we have

$$\Delta(v \cdot \mathbf{a} \cdot \mathbf{b}) = v \cdot \mathbf{a} \otimes 1 + v \otimes \mathbf{b} + \sum_v v_{(1)} \otimes v_{(2)} \cdot \mathbf{a} \cdot \mathbf{b}.$$

Hence by Lemma 5.4 we have

$$\begin{aligned}
\varphi(v \cdot \mathbf{a} \cdot \mathbf{b}) &= \kappa(v \cdot \mathbf{a} \cdot \mathbf{b}) + \varphi(v \cdot \mathbf{a}) \cdot \mathbf{b} \cdot \eta(1) + \varphi(v) \cdot \mathbf{b} \cdot \eta(\mathbf{b}) \\
&\quad + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)} \cdot \mathbf{a} \cdot \mathbf{b}) \\
&= \varphi(v \cdot \mathbf{a}) \cdot \mathbf{b} \cdot 2 + \varphi(v) \cdot \mathbf{b} \cdot 2 \cdot (\mathbf{a} - \mathbf{b}).
\end{aligned}$$

But by Lemma 5.6 we have $\varphi(v \cdot \mathbf{a}) = \varphi(v) \cdot (\mathbf{a} + \mathbf{b})$. So

$$\begin{aligned}\varphi(v \cdot \mathbf{a} \cdot \mathbf{b}) &= \varphi(v) \cdot ((\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} \cdot 2 + \mathbf{b} \cdot 2 \cdot (\mathbf{a} - \mathbf{b})) \\ &= \varphi(v) \cdot 2 \cdot (\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}) \\ &= \varphi(v) \cdot 2\mathbf{d}. \quad \blacksquare\end{aligned}$$

Proof of Proposition 5.5. The proof is by induction on the length of v. It is easy to compute that $\omega(\mathbf{a}) = \mathbf{c} = \varphi(\mathbf{a})$ and $\omega(\mathbf{ab}) = 2\mathbf{d} = \varphi(\mathbf{ab})$.

It follows from the definition of ω that it satisfies the same recursions which are given for φ in Lemmas 5.6 and 5.7. Thus we conclude that $\omega(v) = \varphi(v)$ for all monomials v that begin with the letter $\mathbf{a}$. ▮

Similar to Proposition 5.5, we have for any monomial v that $\varphi(\mathbf{b} \cdot v) = 2\mathbf{b} \cdot \omega(v)$.

6. PROOF OF THE MAIN THEOREM

For an oriented matroid $\mathscr{M}$ let T be the lattice of regions and L be the lattice of flats. Let $\hat{L}$ be the lattice L with a new minimal element $\hat{0}$ adjoined. For a sign vector x define the *zero set* as $z(x) = \{e \in E : x_e = 0\}$. The zero set of a covector of the oriented matroid is a flat in the underlying matroid. Hence by extending the map z by $z(\hat{1}) = \hat{0}$, z is a function from T to $\hat{L}$. We will view z as a function from the dual lattice T^*. Then z is a surjective, order, and rank preserving map from T^* to $\hat{L}$.

Figure 1 illustrates the lattice $\hat{L}$ corresponding to the hyperplane arrangement of the coordinate axes in $\mathbb{R}^2$, together with the dual of the associated lattice of regions. The map z takes elements in the right lattice

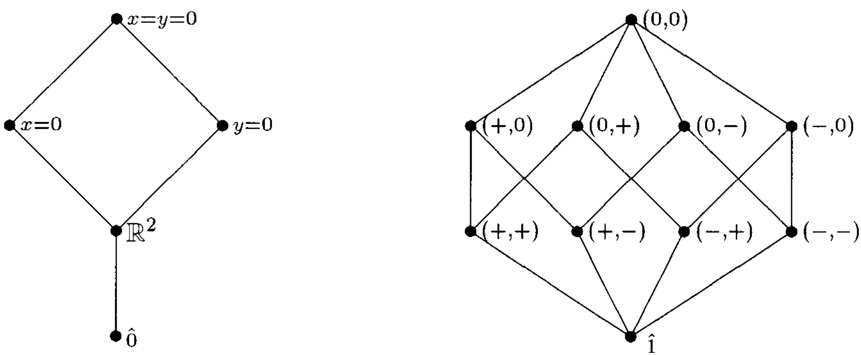

FIG. 1. The lattice $\hat{L}$ and the lattice T^*, the dual of the lattice of regions.

surjectively to elements of the same rank in the left lattice. Here $\Psi(L) = \mathbf{a} + \mathbf{b}$ and $\Psi(T^*) = \omega(\mathbf{a} \cdot \Psi(L)) = \mathbf{c}^2 + 2\mathbf{d}$. This is the **c**-2**d**-index of the square, which is the associated zonotope of this arrangement.

Since z is an order preserving map, z maps a chain from the lattice T^* to a chain in the lattice $\hat{L}$. The following proposition describes the cardinality of the inverse image of a chain in $\hat{L}$; see [7, Proposition 4.6.2].

PROPOSITION 6.1. ([7]). *For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in $\hat{L}$, the cardinality of its inverse image is given by*

$$|z^{-1}(c)| = \prod_{i=1}^{k-1} \sum_{x_i \leqslant y \leqslant x_{i+1}} (-1)^{\rho(x_i, y)} \cdot \mu(x_i, y).$$

Observe that each interval of the form $[x_i, y]$ belongs to L, and hence is a geometric lattice. Recall the sign of the Möbius function of a geometric lattice is positive or negative depending on whether its rank is even or odd, respectively, so $(-1)^{\rho(x_i, y)} \cdot \mu(x_i, y) = |\mu(x_i, y)|$.

Proof of Theorem 3.1. By Lemma 5.2 we may rewrite Proposition 6.1 as

$$\begin{aligned} |z^{-1}(c)| &= \prod_{i=1}^{k-1} 2 \cdot E(\Psi([x_i, x_{i+1}])) \\ &= A(\Psi([x_0, x_1])) \cdot \prod_{i=1}^{k-1} 2 \cdot E(\Psi([x_i, x_{i+1}])), \end{aligned}$$

since $A(\Psi([x_0, x_1])) = 1$, as noted earlier.

Now we can compute the **ab**-index of the lattice T^* by summing over chains c in the lattice $\hat{L}$. Here k denotes the length $l(c)$ of the chain c. The fact that k depends on c will be suppressed in the notation. Observe that we are multiplying non-commutative terms, hence the product $\prod_{i=j}^{k} u_i$ denotes $u_j \cdot u_{j+1} \cdots u_k$. We now have by (3)

$$\begin{aligned} \Psi(T^*) &= \sum_c |z^{-1}(c)| \cdot \mathrm{wt}(c) \\ &= \sum_c A(\Psi([x_0, x_1])) \cdot \prod_{i=1}^{k-1} 2 \cdot E(\Psi([x_i, x_{i+1}])) \cdot \mathrm{wt}(c) \\ &= \sum_c A(\Psi([x_0, x_1])) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_0, x_1) - 1} \\ &\quad \cdot \prod_{i=1}^{k-1} \mathbf{b} \cdot 2 \cdot E(\Psi([x_i, x_{i+1}])) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_i, x_{i+1}) - 1} \\ &= \sum_c \kappa(\Psi([x_0, x_1])) \cdot \prod_{i=1}^{k-1} \mathbf{b} \cdot \eta(\Psi([x_i, x_{i+1}])), \end{aligned}$$

where the last equality follows from equations (5) and (6). Restricting this sum to chains of length k, we may rewrite it in terms of the coproduct on posets. Let $v = \Psi(\hat{L})$. Then

$$\begin{aligned}
\sum_{c:\, l(c)=k} & \kappa(\Psi([x_0, x_1])) \cdot \prod_{i=1}^{k-1} \mathbf{b} \cdot \eta(\Psi([x_i, x_{i+1}])) \\
&= \sum_{\hat{L}} \kappa(\Psi(\hat{L}_{(1)})) \cdot \prod_{i=2}^{k} \mathbf{b} \cdot \eta(\Psi(\hat{L}_{(i)})) \\
&= \sum_{v} \kappa(v_{(1)}) \cdot \prod_{i=2}^{k} \mathbf{b} \cdot \eta(v_{(i)}) \\
&= \varphi_k(v).
\end{aligned}$$

This holds since Ψ is a coalgebra homomorphism; see the discussion after Proposition 4.1. Observe that $\Psi(\hat{L}) = \mathbf{a} \cdot \Psi(L)$, which easily follows from (3), the chain definition of the **ab**-index. If we now sum over all lengths k of chains, we obtain

$$\begin{aligned}
\Psi(T^*) &= \sum_{k \geqslant 1} \varphi_k(\Psi(\hat{L})) \\
&= \varphi(\Psi(\hat{L})) \\
&= \varphi(\mathbf{a} \cdot \Psi(L)) = \omega(\mathbf{a} \cdot \Psi(L)),
\end{aligned}$$

which is the desired expression. ∎

7. APPLICATIONS OF *R*-LABELINGS OF GEOMETRIC LATTICES

When a poset P admits an R-labeling, there is a combinatorial interpretation of $\beta(S)$ and thus of $\Psi(P)$ [6, Theorem 2.7], [23, Theorem 3.13.2]. Stanley showed that every semi-modular lattice admits an R-labeling [22], [23, Example 3.13.5]. This leads to a combinatorial interpretation of the **ab**-index of a geometric lattice and hence an interpretation of the **c**-2**d**-index of the lattice of regions of an oriented matroid. It also enables one to compute these indices in certain cases.

Recall that an *edge-labeling* λ of a locally finite poset P is a map which assigns to each edge in the Hasse diagram of P an element from some poset Λ. For us Λ will always be a linearly ordered set. If y covers x in P, then we denote the label on the edge (x, y) by $\lambda(x, y)$. A maximal chain $x = x_0 < x_1 < \cdots < x_k = y$ in an interval $[x, y]$ in P is called *rising* if the labels are weakly increasing with respect to the order of the poset Λ, that is, $\lambda(x_0, x_1) \leqslant_\Lambda \lambda(x_1, x_2) \leqslant_\Lambda \cdots \leqslant_\Lambda \lambda(x_{k-1}, x_k)$. An edge-labeling is called an *R-labeling* if for every interval $[x, y]$ in P there is a unique rising maximal chain in $[x, y]$.

Let P be a poset of rank $n+1$ with R-labeling λ. For a maximal chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{1}\}$ let $\lambda(c)$ denote its list of labels, that is, $\lambda(c) = (\lambda(x_0, x_1), ..., \lambda(x_n, x_{n+1}))$. Let the descent set of a list of labels $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be the set $D(\lambda) = \{i : \lambda_i >_A \lambda_{i+1}\}$. Observe that $D(\lambda)$ is a subset of the set $\{1, ..., n\}$. We also let the descent of a maximal chain c, $D(c)$, be the set $D(\lambda(c))$.

Our interest in R-labelings stems from the following result of Björner and Stanley; see [6, Theorem 2.7]:

PROPOSITION 7.1. *Let P be a graded poset that admits an R-labeling. Then $\beta(S)$ is equal to the number of maximal chains c with descent set S.*

From this result we obtain the following corollary, which was observed in [10].

COROLLARY 7.2. *Let P be a graded poset of rank $n+1$. Let λ be an R-labeling of P, and let M be the multiset of all lists of labels of maximal chains of P. Then the* **ab**-*index of P is given by*

$$\Psi(P) = \sum_{\lambda \in M} u_{D(\lambda)} = \sum_{c} u_{D(c)},$$

where the second sum is over all maximal chains c of the poset P.

An R-labeling of a geometric lattice L can be obtained as follows; see [23, Example 3.13.5]. Let A denote the set of atoms of L and let there be a total ordering on the atoms. Let the label on the edge $x < y$ be described by

$$\lambda(x, y) = \min\{a \in A : x \vee a = y\}.$$

Observe that with this R-labeling two different chains will have two different lists of labels. Hence the multiset M of lists of labels is really a set in this case.

For a list of labels $\lambda = (\lambda_1, ..., \lambda_{n+1})$, call a position i a *peak* if $\lambda_{i-1} <_A \lambda_i >_A \lambda_{i+1}$. Observe that a peak corresponds to the factor **ab** in the **ab**-monomial $u_{D(\lambda)}$. Thus $\omega(\mathbf{a} \cdot u_{D(\lambda)})$ may be computed by augmenting the list of labels λ with an initial label λ_0 defined to be smaller than any other label, then assigning the weight 2**d** to the peaks in the list and the weight **c** to the other positions $i \leqslant n$. As an example, we have that if $\lambda = (2, 5, 3, 1, 4, 7, 6)$, then $\omega(\mathbf{a} \cdot u_{D(\lambda)}) = \mathbf{c} \cdot 2\mathbf{d} \cdot \mathbf{c}^2 \cdot 2\mathbf{d}$, since there are peaks at the 2nd and the 6th positions in λ, reading from left to right.

From Theorem 3.1 and Corollary 7.2 we have the following corollary.

COROLLARY 7.3. *Let $\mathcal{M}$ be an oriented matroid, T its lattice of regions, and L its lattice of flats. Let M be the set of label lists of the geometric lattice L. Then the* **c**-2**d**-*index of T^* is given by*

$$\Psi(T)^* = \sum_{\lambda \in M} \omega(\mathbf{a} \cdot u_{D(\lambda)}).$$

PROPOSITION 7.4. *Let L be a geometric lattice of rank n. Then for all $S \subseteq \{1, ..., n-1\}$ we have $\beta_L(S) \geqslant \beta_{B_n}(S)$. Hence the* **ab**-*index $\Psi(L)$ is coefficient-wise greater than or equal to the* **ab**-*index of the Boolean algebra B_n.*

Proof. The geometric lattice L corresponds to a matroid on the set of atoms. Suppose that the atoms $a_1, ..., a_n$ form a base for this matroid. Then we have that the subposet P of L consisting of all elements of the form $a_I = \bigvee_{i \in I} a_i$, where $I \subseteq \{1, ..., n\}$, is isomorphic to the Boolean algebra B_n.

Choose a linear order on the atoms of L so that the atoms $a_1 < a_2 < \cdots < a_n$ form an initial segment in the order. We now have an R-labeling of the geometric lattice L that corresponds to this linear order. This R-labeling has the property that if we restrict our attention to the subposet P, the labels in P is the standard R-labeling of the Boolean algebra.

Let S be a subset of $\{1, ..., n-1\}$. Now we have $\beta_L(S)$ is the number of maximal chains in L with descent set S. This set of maximal chains contains all maximal chains in P with descent set S. The number of such chains is $\beta_P(S) = \beta_{B_n}(S)$, and so $\beta_{B_n}(S) \leqslant \beta_L(S)$. ∎

The hyperplane arrangement $\{\mathbf{x} \in \mathbb{R}^n : x_i = 0\}$ where $i = 1, ..., n$, has the lattice of flats (the intersection lattice) to be the Boolean algebra B_n. Moreover, the corresponding zonotope is the n-dimensional cube. Hence by combining Theorem 3.1 and Proposition 7.4, we have the following interesting corollary.

COROLLARY 7.5. *Let T be the lattice of regions of an oriented matroid $\mathcal{M}$ of rank n. Then the* **c**-2**d**-*index $\Psi(T)$ is coefficient-wise greater than or equal to the* **c**-2**d**-*index of the dual of the cubical lattice, $\Psi(C_n)^*$.*

COROLLARY 7.6. *Among all zonotopes of dimension n, the n-dimensional cube has the smallest* **c**-2**d**-*index.*

We may view this corollary as the analogue for zonotopes of the following conjecture:

CONJECTURE 7.7 (Stanley [25]). *The* **cd**-*index of a convex polytope is coefficient-wise greater than or equal to the* **cd**-*index of the simplex of the same dimension. More generally, the* **cd**-*index of a Gorenstein* lattice is* *coefficient-wise greater than or equal to the* **cd**-*index of the Boolean algebra of the same rank.*

8. THE CUBICAL LATTICE AND THE BOOLEAN ALGEBRA

In this section we will apply our results to the cubical lattice. As a consequence we will obtain new formulas for the **c**-2**d**-index of the cubical lattice and the **cd**-index of the Boolean algebra. These identities will imply results for simsun and André permutations.

Recall that the cubical lattice C_n is the face lattice of a zonotope (the n-cube). The corresponding lattice of flats is the Boolean algebra B_n. The labels of a maximal chain in B_n is a permutation of the elements 1, 2, ..., n. Let S_n denote the symmetric group on n elements. Hence by Corollary 7.3 we have the following proposition.

PROPOSITION 8.1. *The* **c**-2**d**-*index of the cubical lattice* C_n *is given by*

$$\Psi(C_n) = \sum_{\pi \in S_n} \omega(\mathbf{a} \cdot u_{D(\pi)}).$$

For instance, when $n = 3$ we have

π	$\omega(\mathbf{a} \cdot u_{D(\pi)})$	π	$\omega(\mathbf{a} \cdot u_{D(\pi)})$
(1, 2, 3)	$\mathbf{c}^3$	(2, 3, 1)	$\mathbf{c} \cdot 2\mathbf{d}$
(1, 3, 2)	$\mathbf{c} \cdot 2\mathbf{d}$	(3, 1, 2)	$2\mathbf{d} \cdot \mathbf{c}$
(2, 1, 3)	$2\mathbf{d} \cdot \mathbf{c}$	(3, 2, 1)	$2\mathbf{d} \cdot \mathbf{c}$

So $\Psi(C_3) = \mathbf{c}^3 + 2 \cdot \mathbf{c} \cdot 2\mathbf{d} + 3 \cdot 2\mathbf{d} \cdot \mathbf{c} = \mathbf{c}^3 + 4 \cdot \mathbf{cd} + 6 \cdot \mathbf{dc}$.

Proposition 8.1 gives an explicit combinatorial interpretation for the **c**-2**d**-index of the n-dimensional cube. Using this interpretation we find a similar combinatorial interpretation for the **cd**-index of the simplex.

PROPOSITION 8.2. *The* **cd**-*index of the Boolean algebra* B_n *is given by*

$$\Psi(B_n) = \frac{1}{2^{n-1}} \cdot \sum_{\pi \in S_n} \omega(u_{D(\pi)}).$$

Proof. Recall that $\mathscr{P}$ is the integer span of all graded posets. Define a linear function $H : \mathscr{P} \to \mathscr{P}$ by $H(B_1) = 0$ and for a poset P of rank greater than or equal to 2 by

$$H(P) = \sum_{a} [a, \hat{1}],$$

where the sum ranges over all atoms a of the poset P. Similarly, define a linear function $h\colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by $h(1)=0$, $h(\mathbf{a}\cdot u)=u$, and $h(\mathbf{b}\cdot u)=u$. By the chain definition (3) of the **ab**-index we obtain that $\Psi(H(P))=h(\Psi(P))$ for all posets P.

Observe that $h(\mathbf{c}\cdot u)=2\cdot u$ and $h(\mathbf{d}\cdot u)=\mathbf{c}\cdot u$. Hence the linear function h on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ restricts to a linear function from **cd**-polynomials to **cd**-polynomials. Moreover, by considering the three cases $v=1$, $v=\mathbf{a}\cdot u$, and $v=\mathbf{b}\cdot u$, it is now easy to prove

$$h(\omega(\mathbf{a}\cdot v))=2\cdot\omega(v).$$

Observe that $H(C_n)=2^n\cdot B_n$, since the cubical lattice of rank $n+1$ has 2^n atoms and each interval $[a, \hat{1}]$ in C_n is isomorphic to B_n. Applying this relation to Proposition 8.1, we obtain

$$\begin{aligned}
2^n\cdot\Psi(B_n) &= h(\Psi(C_n))\\
&= \sum_{\pi\in S_n} h(\omega(\mathbf{a}\cdot u_{D(\pi)}))\\
&= \sum_{\pi\in S_n} 2\cdot\omega(u_{D(\pi)}),
\end{aligned}$$

which completes the proof. ∎

Propositions 8.1 and 8.2 give an explicit way to compute the **cd**-index of the Boolean algebra and the cubical lattice. Earlier expressions of these **cd**-indexes has involved André and simsun permutations, and their corresponding signed versions. Hence Propositions 8.1 and 8.2 can be translated into results about these classes of permutations and their descent sets.

To avoid being lengthy, we refer the reader to the literature for the definitions of these permutation classes and their relation to the **cd**-index: for simsun permutations, see [24, 27]; for André permutations, see [21]; for signed André permutations, see [10, 21]; and for signed simsun permutations, see [11].

PROPOSITION 8.3. *The number of simsun permutations in S_{n-1} with descent set S and the number of André permutations in S_{n-1} with descent set S is equal to $2^{|S|-n+1}$ times the number of permutations in S_n with peaks at the positions S shifted up by one.*

This proposition generalizes the first equation on page 129 in [14]. Let $S_n^{\pm}$ denote the group of signed permutations on n elements.

PROPOSITION 8.4. *The number of signed simsun permutations in $S_{n-1}^{\pm}$ with descent set S and the number of signed André permutations in $S_{n-1}^{\pm}$ with* *descent set S is equal to $2^{|S|}$ times the number of permutations in S_n with peaks at positions S.*

Let E_n denote the nth Euler number, that is, E_n is the number of alternating permutations in S_n that begins with an ascent. It is well-known that $\sum_{n \geqslant 0} E_n x^n/n! = \sec(x) + \tan(x)$. Now as two corollaries we obtain:

Corollary 8.5. *The number of simsun permutations in S_{2k} with k descents and the number of André permutations in S_{2k} with k descents is equal to $2^{-k} \cdot E_{2k+1}$. This is the coefficient of $\mathbf{d}^k$ in $\Psi(B_{2k+1})$.*

Corollary 8.6. *The number of signed simsun permutations in $S_{2k}^{\pm}$ with k descents and the number of signed André permutations in $S_{2k}^{\pm}$ with k descents is equal to $2^k \cdot E_{2k}$. This is the coefficient of $\mathbf{d}^k$ in $\Psi(C_{2k})$.*

The first part of Corollary 8.5 is due to Sundaram [27, Proposition 1.6]. The second part of Corollary 8.6 also follows from Corollary 3.4

9. COMPUTATION OF $\Psi(T)^*$ FROM THE $\alpha_L(S)$'S

Let T be the lattice of regions of an oriented matroid and let L be the lattice of flats. Assume we know all the values of $\alpha_L(S)$. Then by Eq. (1) we may compute $\beta_L(S)$, and thus $\Psi(L)$. Theorem 3.1 allows us to compute the **c**-2**d**-index of the Eulerian poset T. As an example, consider the case when L has rank 4. The coefficients of the **c**-2**d**-index $\Psi(T)^*$ are given by:

$$\begin{aligned}
\mathbf{c}^4 &: \alpha(\varnothing),\\
\mathbf{c}^2 2\mathbf{d} &: \alpha(3) - \alpha(\varnothing),\\
\mathbf{c}2\mathbf{d}\mathbf{c} &: \alpha(2, 3) - \alpha(3),\\
2\mathbf{d}\mathbf{c}^2 &: \alpha(1, 2, 3) - \alpha(1, 3) - \alpha(2, 3) + \alpha(3) + \alpha(1) - \alpha(\varnothing),\\
(2\mathbf{d})^2 &: \alpha(1, 3) - \alpha(1) - \alpha(3) + \alpha(\varnothing).
\end{aligned}$$

Observe that $\alpha(2)$ and $\alpha(1, 2)$ do not occur in the coefficients of the **c**-2**d**-index. That is, of the $2^3 = 8$ values of $\alpha(S)$, we only need 6 of them to compute $\Psi(T)$.

The main result in this section is to demonstrate which of the $\alpha(S)$ are needed in the computation of $\Psi(T)$. In order to do this, let $\mathscr{P}_n$ be the collection of subsets of $\{1, ..., n\}$ described by

$$\mathscr{P}_n = \{\varnothing\} \cup \{S \subseteq \{1, ..., n\} : S \neq \varnothing, \max(S) \equiv n \bmod 2\}.$$

THEOREM 9.1. *When L has* rank $n+1$, *we only need* $\alpha_L(S)$, *where* $S \in \mathscr{P}_n$, *to compute* $\Psi(T)$.

This theorem is a generalization of [30, Corollary 5.5], where Zaslavsky essentially shows this for the f-vector of the lattice of regions.

As an example, consider again the case when $n=3$. The collection $\mathscr{P}_3$ consists of all subsets of $\{1, 2, 3\}$ with the rank of the maximal element having the same parity as $n=3$. These subsets are:

$$\mathscr{P}_3 = \{\varnothing, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

which are exactly the sets that appeared in the previous calculations.

The following lemma will be useful to us in the proof of Theorem 9.1. We may view it as a generalization of Eq. (1) and the inverse relation of Eq. (1), namely the identity $\alpha(S) = \sum_{T \subseteq S} \beta(T)$.

LEMMA 9.2. *Let S be a subset of $\{1, ..., n\}$ and let V be a subset of S. Let $\bar{V}$ denote the complement of the set V in the set S, that is, the set $S - V$. Then we have*

$$\sum_{V \subseteq T \subseteq S} \beta(T) = \sum_{\bar{V} \subseteq T \subseteq S} (-1)^{|S-T|} \cdot \alpha(T).$$

Proof. The proof is a direct computation.

$$\begin{aligned}
\sum_{V \subseteq T \subseteq S} \beta(T) &= \sum_{V \subseteq T \subseteq S} \sum_{R \subseteq T} (-1)^{|T-R|} \cdot \alpha(R) \\
&= \sum_{R \subseteq S} \left(\sum_{R \cup V \subseteq T \subseteq S} (-1)^{|T-R|} \right) \cdot \alpha(R) \\
&= \sum_{R \subseteq S} \chi(R \cup V = S) \cdot (-1)^{|S-R|} \cdot \alpha(R) \\
&= \sum_{\bar{V} \subseteq R \subseteq S} (-1)^{|S-R|} \cdot \alpha(R). \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 9.1. First we will prove that $\alpha(S)$, where S belongs to $\mathscr{P}_n$, is necessary to compute $\Psi(T)^*$. The coefficient of $\mathbf{c}^{n+1}$ is $\alpha(\varnothing)$, so the empty set is necessary. Consider the $\mathbf{c}$-$2\mathbf{d}$-monomial $w = 2\mathbf{d}\mathbf{c}^{n-1}$. The only $\mathbf{ab}$-monomials v such that $\omega(\mathbf{a} \cdot v) = w$ have the form $\mathbf{b}^i\mathbf{a}^{n-i}$, where $1 \leqslant i \leqslant n$. Hence the coefficient of w in $\Psi(T)^*$ is given by the sum $\sum_{i=1}^{n} \beta(1, ..., i)$. By Lemma 9.2, when n is even this sum can be written as

$$\sum_{j=1}^{n/2} (\beta(1, ..., 2j-1) + \beta(1, ..., 2j)) = \sum_{j=1}^{n/2} \sum_{2j \in T \subseteq \{1, ..., 2j\}} (-1)^{|T|} \cdot \alpha(T).$$

Observe that every non-empty set of $\mathscr{P}_n$ occurs in this double sum. There is a similar expansion in the case when n is odd. This shows that the sets in $\mathscr{P}_n$ are necessary in order to compute $\Psi(T)^*$.

Let w be the **c**-2**d**-monomial $w = \mathbf{c}^{k_0} \cdot 2\mathbf{d} \cdot \mathbf{c}^{k_1} \cdot 2\mathbf{d} \cdots 2\mathbf{d} \cdot \mathbf{c}^{k_r}$ of degree $n+1$ and let C be the coefficient of w in $\Psi(T^*)$. Let P be the index set $\{0, ..., k_1\} \times \cdots \times \{0, ..., k_r\}$. For $p = (p_1, ..., p_r)$ in P define the **ab**-monomial $v(p)$ by

$$v(p) = \mathbf{a}^{k_0}\mathbf{b} \cdot \mathbf{b}^{p_1}\mathbf{a}^{k_1 - p_1} \cdot \mathbf{ab} \cdot \mathbf{b}^{p_2}\mathbf{a}^{k_2 - p_2} \cdot \mathbf{ab} \cdots \mathbf{ab} \cdot \mathbf{b}^{p_r}\mathbf{a}^{k_r - p_r}.$$

We have now $\omega(\mathbf{a} \cdot v(p)) = w$. The converse is also true: if v is an **ab**-monomial so that $\omega(\mathbf{a} \cdot v) = w$, then v is equal to $v(p)$ for some $p \in P$. Hence C is the sum of all the coefficients of the **ab**-monomials $v(p)$ in the **ab**-index $\Psi(L)$.

Let $K_i = k_0 + \cdots + k_{i-1} + 2 \cdot i - 1$. We then have $K_r + k_r = n$. Define the set $s(p)$ by

$$\{K_1, ..., K_1 + p_1, K_2, ..., K_2 + p_2, ..., K_r, ..., K_r + p_r\}.$$

It is now easy to verify that $u_{s(p)} = v(p)$.

Consider the set $\{0, 1, ..., k_i\}$. We will partition this set into $h_i = \lceil (k_i + 1)/2 \rceil$ smaller sets. When k_i is odd we use the partition $\{0, 1\} \cup \{2, 3\} \cup \cdots \cup \{k_i - 1, k_i\}$, while when k_i is even we use $\{0\} \cup \{1, 2\} \cup \cdots \cup \{k_i - 1, k_i\}$. Formally, we write this as

$$\{0, 1, ..., k_i\} = \{n_{i,1}, m_{i,1}\} \cup \{n_{i,2}, m_{i,2}\} \cup \cdots \cup \{n_{i,h_i}, m_{i,h_i}\}.$$

The following two observations will be used later. First, $m_{i,j}$ is equivalent to k_i modulo 2. Second, $m_{i,j}$ and $n_{i,j}$ are equal if and only if $j = 1$ and k_i is even. In this case we have that $m_{i,j} = n_{i,j} = 0$.

We now obtain a partition of the index set P into $h_1 \times \cdots \times h_r$ pieces by considering each partition of $\{0, 1, ..., k_i\}$ component-wise. Let Q be the index set $\{1, ..., h_1\} \times \cdots \times \{1, ..., h_r\}$. For $q = (q_1, ..., q_r) \in Q$, let $R(q)$ be the subset $R(q) = \{p \in P : p_i \in \{n_{i,q_i}, m_{i,q_i}\}\}$. Thus the coefficient C is given by the sum

$$C = \sum_{p \in P} \beta(s(p)) = \sum_{q \in Q} \sum_{p \in R(q)} \beta(s(p)).$$

Consider an element q in the index set Q. We will prove that when we expand the sum $\sum_{p \in R(q)} \beta(s(p))$ in terms of $\alpha(T)$'s we only obtain sets that belong to $\mathscr{P}_n$. For ease in notation, let $m_i = m_{i,q_i}$ and $n_i = n_{i,q_i}$. Let S be the set $s(m_1, ..., m_r)$ and V be the set $s(n_1, ..., n_r)$. The set V is contained in S and $\bar{V} = \{K_i + m_i : 1 \leqslant i \leqslant r, n_i \neq m_i\}$. Observe that summing over the elements in $R(q)$ corresponds to summing over the interval $V \subseteq T \subseteq S$ in the Boolean algebra. Hence we have by Lemma 9.2 that

$$\begin{aligned}\sum_{p \in R(q)} \beta(s(p)) &= \sum_{V \subseteq T \subseteq S} \beta(T)\\ &= \sum_{\bar{V} \subseteq T \subseteq S} (-1)^{|S-T|} \cdot \alpha(T).\end{aligned}$$

Let i be the index such that $K_i + m_i = \max(\bar{V})$. If $\bar{V}$ is the empty set we let $i = 0$. Hence, for $i+1 \leqslant j \leqslant r$ we have that $n_j = m_j$. This implies that $m_j = 0$ and that k_j is even. Now we obtain $n = K_r + k_r \equiv K_r \bmod 2$ and that for $i+1 \leqslant j \leqslant r-1$ we have $K_{j+1} = K_j + k_j + 2 \equiv K_j \bmod 2$. Hence $n \equiv K_r \equiv \cdots \equiv K_{i+1} \bmod 2$. Moreover, if $\bar{V}$ is non-empty, we have $K_{i+1} = K_i + k_i + 2 \equiv K_i + m_i \equiv \max(\bar{V}) \bmod 2$.

The set $\{s \in S : s > \max(\bar{V})\}$ has the form $\{K_{i+1}, K_{i+2}, ..., K_r\}$. Consider a non-empty set T such that $\bar{V} \subseteq T \subseteq S$. We have that the maximal element of T belongs to the set $\{\max(\bar{V}), K_{i+1}, ..., K_r\}$. But all the numbers in this set are congruent to n modulo 2. This completes the proof. ∎

COROLLARY 9.3. *If $\alpha(T)$ occurs in the expansion of the coefficient of the* **c**-2**d**-*monomial* $w = \mathbf{c}^{k_0} \cdot 2\mathbf{d} \cdot \mathbf{c}^{k_1} \cdot 2\mathbf{d} \cdots 2\mathbf{d} \cdot \mathbf{c}^{k_r}$ *of degree $n+1$, then the coefficient of $\alpha(T)$ is given by*

$$(-1)^{n-k_0-r+1-|T|}.$$

Proof. We first claim that the term $\alpha(T)$ corresponds to at most one entry in the index set Q, as defined in the proof of Theorem 9.1. Assume on the contrary that $\alpha(T)$ appears in two different pieces, say q and q'. Following the notation of the proof of Theorem 9.1, we have that $\bar{V} \subseteq T \subseteq S$ and $\overline{V'} \subseteq T \subseteq S'$. But since q and q' are different, without loss of generality there is an index i such that $n'_i \leqslant m'_i < n_i < m_i$. We now have that $m_i \in S$ and $m_i \notin V$. This implies that m_i belongs to the set T. Since m_i does not belong the set S', we obtain our desired contradiction. Hence we know that no cancellation will occur in the expansion given in the proof of the theorem.

The coefficient of $\alpha(T)$ in the expansion is given by $(-1)^{|S|-|T|}$. The cardinality of the set S is $(m_1+1) + \cdots + (m_r+1)$, which is equivalent to $k_1 + \cdots + k_r + r$ modulo 2. This is equal to $K_r - k_0 - r + 1 + k_r = n - k_0 - r - 1$, which proves the corollary. ∎

LEMMA 9.4. *The cardinality of $\mathscr{P}_n$ is $\lceil 2 \cdot 2^n/3 \rceil$.*

Proof. For $k \equiv n \bmod 2$ we have that the number of sets in $\mathscr{P}_n$ with k as the maximal element is 2^{k-1}. Hence the cardinality of $\mathscr{P}_n$ is given by the sum

$$1 + 2^{n-1} + 2^{n-3} + \cdots + (2 \text{ or } 1).$$

When n is even this evaluates to

$$1 + \frac{2^{n+1} - 2}{4 - 1} = \frac{2^{n+1} + 1}{3} = \left\lceil \frac{2^{n+1} + 1}{3} - \frac{1}{3} \right\rceil = \left\lceil \frac{2 \cdot 2^n}{3} \right\rceil.$$

When n is odd, a similar computation yields the same result. ∎

The results in this section suggest the following method to compute $\Psi(T)$ from $\alpha(S)$ where $S \in \mathscr{P}_n$. Let $\alpha'(S) = \alpha(S)$ if $S \in \mathscr{P}_n$ and otherwise let $\alpha'(S) = 0$. Analogous to Eq. (1) let

$$\beta'(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \alpha'(T).$$

Let $\Psi'(L) = \sum_S \beta'(S) \cdot u_S$. Then we obtain that $\Psi(T) = \omega(\mathbf{a} \cdot \Psi'(L))$.

10. CONCLUDING REMARKS

In a forthcoming paper the authors use the techniques developed in Section 7 to give an explicit recursion for the **c**-2**d**-index of the braid arrangements of types A_n and B_n. Recall the two zonotopes corresponding to these two braid arrangements are the permutahedron and the signed permutahedron. Similar recursions for other such families of arrangements might be of interest.

In another forthcoming paper, the authors show that the $\alpha_L(S)$ for a geometric lattice L satisfy no linear relations. Thus fully a third of the information in $\Psi(L)$ is not needed for the computation of the **c**-2**d**-index $\Psi(T)$ of its lattice of regions. On the other hand, linear inequalities on the coefficients of the **c**-2**d**-index of T, derived from those known to hold for all zonotopes or polytopes, imply linear relations on the $\alpha_L(S)$ that hold for all *orientable* geometric lattices L, at least those realizable over the reals. A question currently under investigation is whether such inequalities might shed some light on the conjectured unimodality of the Whitney numbers of the first and second kind for these subclasses of geometric lattices.

In the case of non-orientable geometric lattices L, one still can define the **c**-2**d**-index $\Psi(T)$, although there is no associated lattice of regions T. In this case, one can ask whether the coefficients of $\Psi(T)$ have any meaning for L.

Finally, by using the Foata–Strehl group action on the symmetric group, [14], one may prove Proposition 8.3. This would be a bijective proof of the proposition. Is there a similar group action on the group of signed permutations such that one would obtain a bijective proof of the results in Proposition 8.4?

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